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Basic Astronomical Estimates

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Abstract

We present astronomical estimates about the Earth, Moon, Sun and the nearest fixed stars, in particular of sizes, distances and masses. Our focus is on the question to which extent those were already known in antiquity.

1 Introduction

Throughout history astronomers have strived to determine the dimensions of the Universe. Since the Babylonian era various estimates about the Earth, Moon, Sun and stars have been calculated, assisted by empirical observations. These calculations have been corrected throughout time and improved with technology and increased knowledge of the Universe. In what follows, we will go through a list of astronomical objects one by one beginning with our own planet.

2 The Earth

2.1 The Earth's Circumference

The Earth's circumference around the equator is 40075.017 km [1]. This is today's accepted most precise value obtained through substantial technological efforts, but surprisingly accurate estimates can already be found by rather simple means, using, for instance, just a metre stick, a stop watch and a sunrise [2]. In the distant past, however, there have been just a few men who were able to measure the circumference accurately. In this section the focus will be on Eratosthenes and Biruni, two mathematicians who used two different methods in different eras to obtain the measurement.

Pythagoras (6th century BC) was one of the first to believe the Earth was a sphere and his theory was supported by Aristotle (4th century BC) the leading scientific authority of his age. Soon after, efforts were undertaken to determine the Earth's circumference. The first real attempt was made by Greek scholar Eratosthenes (276-195 BC), in Egypt. He noticed that on the summer solstice the sun was directly overhead at Syene and not at Alexandria. Figure 1 illustrates his method:

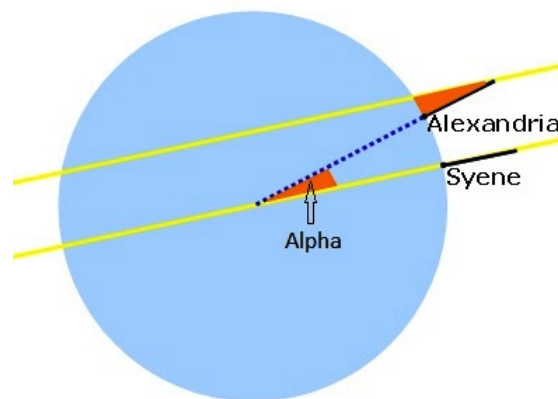


Figure 1: Eratosthenes' Method [3].

Eratosthenes measured the angle α by observing the shadow left by the vertical at Alexandria. Using the known distance between the two settlements he rearranged the formula

$$\frac{\alpha}{360} = \frac{\text{distance}}{\text{circumference}} \quad (1)$$

to obtain the circumference. It is not known exactly how accurate his estimate was, as it depends on what type of stadion (the then common, but geographically varying, unit of distance) he was using. If he used the Attic stadion (185 m) his method gives 46600 km giving an error of 15 %; however if the Egyptian stadion (157.5 m) was used the calculations give 39690 km which amounts to an error of about 1 % in comparison with today's accepted value [4]. For the radius r of the Earth the latter assumption implies a value of

$$r = 6320 \text{ km} , \quad (2)$$

again fairly close to the modern value for the equatorial radius of 6378 km [4].

Abu Rayhan Biruni (AD 973-1048) developed new trigonometric calculations based on the angle between a plain and a mountain top to find the Earth's radius. Figure 2 shows his method:

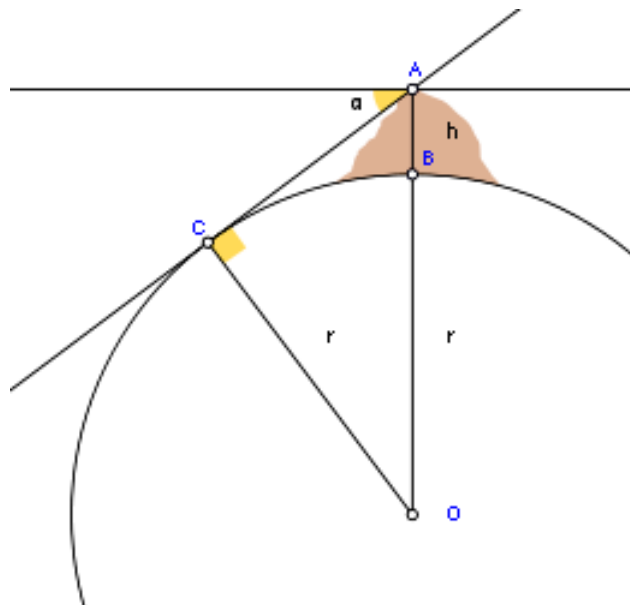


Figure 2: Biruni's Method [5].

Biruni first found the height h of the mountain by going to two points at sea level with a known distance d apart and then measuring the angle between the plain and mountain top for both points (θ and ϕ). Using basic trigonometry for the tangents one finds two equations for the unknown height of and distance to the mountain.

Eliminating the latter results in a formula for the mountain height,

$$h = \frac{\tan(\theta) \tan(\phi)}{\tan(\phi) - \tan(\theta)} d, \quad (3)$$

in terms of the distance d between the points of observation. By standing at the top of the mountain Biruni then measured the dip angle α (see Fig. 2) using an astrolabe. Again, basic trigonometry for the right triangle of Fig. 2 determines the unknown radius of the Earth via

$$r = \frac{h \cos(\alpha)}{1 - \cos(\alpha)}, \quad (4)$$

with the height h given in (3). From his measurements Biruni found the radius r to be 6340 km which gives a circumference of 39800 km. This value differs by only 0.6 % from today's accepted value and shows his method to be very accurate.

3 The Moon

3.1 Distance to the Moon

Ancient civilisations dating back as far as the Babylonians have studied the Moon, noting its phases and using the cycles of New Moon and Full Moon to define the months. However, it wasn't until 270 BC that Greek astronomer Aristarchus (310-230 BC) found an equation to derive the distance from the Earth to the Moon. By studying the duration t of a lunar eclipse, he assumed that if the Moon moved around the Earth at a constant speed, it would take time T to cover a whole orbit of length $2\pi R$ and time t to cover the width of the Earth's shadow, $2r$ (where R is the radius of the Moon's orbit and r is the radius of the earth) [6]. Constancy of the Moon's speed implies that sizes and times are in proportion according to

$$\frac{2\pi R}{2r} = \frac{T}{t}. \quad (5)$$

By measuring the duration t Aristarchus found that $R/r \approx 60$ implying a distance from the Earth to the Moon of approximately 60 Earth radii. If we substitute Eratosthenes' figure (2) for Earth's radius into this, we find an Earth-Moon distance of about 380,000 km. Using today's modern methods, the mean distance between the Earth and Moon is accepted to be 384,400 km [7], which shows that Aristarchus' result, $R = 60r$, is surprisingly accurate.

In our 'space age' the distance between the Earth and the Moon has been determined to a high precision with an error of just a few millimetres. Apollo astronauts visiting the Moon left behind specialised mirrors on its surface. Scientists pointed lasers at the mirrors on the Moon and measured the time it took for the lasers to return to Earth [8]. The accurately known speed of light then yields the distance.

Once the distance from the Earth to the Moon is known, the diameter of the Moon's orbit around Earth is simply twice this distance (assuming a circular orbit). Aristarchus used this to determine the length L of the Moon's orbit, using the simple calculation that the circumference is given by π multiplied by the diameter, $L = 2\pi R \approx 2.4 \times 10^6$ km.

3.2 Diameter of the Moon

Around the same time that he found a value for the distance to the Moon, Aristarchus also calculated the diameter S of the Moon. By observing a lunar eclipse in 270 BC, he found that the Moon moved across the sky at an amount equal to 2.5 times the Moon's angular diameter. He also noted that the Earth's shadow on the Moon lost a distance equal to that of the Moon's diameter [9]. Hence, we obtain the equations

$$2.5 S = 2r - S, \quad \text{or} \quad S = \frac{4}{7} r. \quad (6)$$

Substituting the value (2) for the Earth radius once again, one finds the diameter of the Moon to be 3610 km. The value of the Moon's diameter accepted today is 3476 km [10], which again shows the ingenuity of Aristarchus' geometrical methods.

4 The Sun

4.1 Distance from the Earth to the Sun

Hipparchus (190-120 BC) estimated the distance from the Earth to the Sun to be approximately 500 Earth radii. Claudius Ptolemy (AD 90-168), a Roman mathematician living in Egypt, used the method developed by Hipparchus to deduce that the Sun is about 1200 earth radii from the Earth. This estimate was accepted until Johannes Kepler argued for a much larger distance in the 17th century.

It is now known that the distance from the Earth to the Sun is 1.5×10^8 km (about 230,000 earth radii), 460 times the original estimate by Hipparchus. The mean distance from the Earth to the Sun is called an astronomical unit (1 AU). It can be found by measuring the distance from the Earth to other planets (e.g. Venus) relative to the Sun's distance, as shown in Fig. 3.

By astronomical observations, one can determine when Venus is farthest from the Sun, at the point of its greatest elongation, and hence measure the angle ϵ between the Sun and Venus. The distance d_{EV} from Earth to Venus can be measured by transmitting a radio wave from Earth and recording the time taken for the pulse to reflect off Venus and return to Earth. Since radio waves travel at the speed of light, it

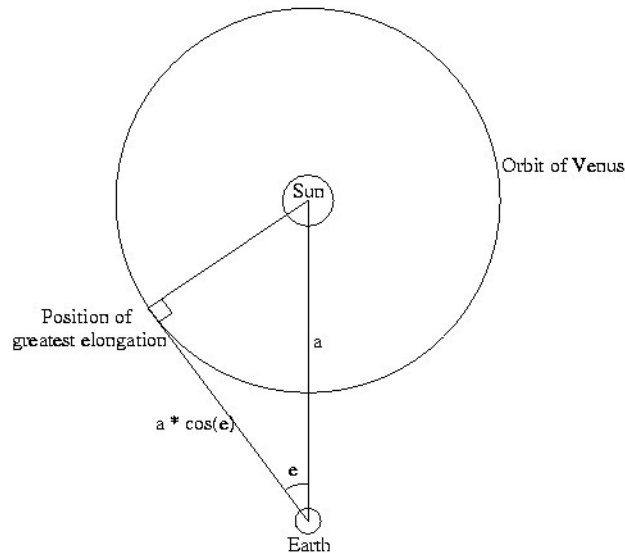


Figure 3: Distance from Earth to Sun [11].

is simple then to calculate the distance from the Earth to the Sun, a ,

$$a = \frac{d_{EV}}{\cos(e)} . \quad (7)$$

4.2 Diameter of the Sun

Today, the diameter of the Sun is estimated at 1.4×10^6 km, approximately 109 times the diameter of the Earth. The volume of the Sun is estimated at 1.4×10^{16} km³, approximately 1.3×10^6 times that of the Earth [12]. These known dimensions are the result of a continuous increase during the Sun's life and will continue to grow over billions of years until the Sun becomes a red giant, roughly 700 times bigger and 1.4×10^4 times brighter. Though the Sun will increase in size, both its mass and temperature will decrease.

It is simple to estimate the diameter of the Sun by using a pin-hole camera and the result (7) [13]. Let d denote the diameter of the projected image of the Sun and l the length of the pinhole camera. The actual diameter D of the Sun is then found from the statement of equal proportions,

$$D/d = a/l . \quad (8)$$

5 Distance to Stars

5.1 Methodology

The method used for the first determination of the distance to a star, a method used to this day, is called trigonometric parallax. By observing stars at different orbital positions of the Earth, astronomers can record the shift of nearby stars, relative to more distant stars. The shift of nearby stars and the new position of the Earth allow astronomers to use geometry to record the parallax angle and to discover the distance to nearby stars.

The parallax (symbolized by the Greek letter, θ) is defined as the angular size of an elliptical arc that the star seems to trace against the background of space during a quarter-period of the Earth's orbit,

$$\tan(\theta) = \frac{a}{d}, \quad (9)$$

where a is the radius of the Earth's orbit (equal to 1 AU, see Section 4.1), and d is the distance to the star (Fig. 4).

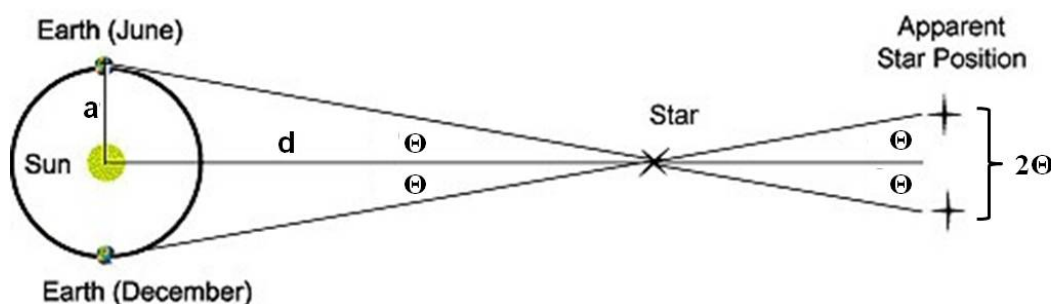


Figure 4: Trigonometric Parallax [14].

5.2 Some History

Alpha Centauri was discovered in the 1800's and became the first star to have its distance measured in 1833 [15, 16] by Thomas Henderson (1798-1844). However, as Henderson had doubts about the accuracy of his instruments he did not publish his work until 1839, a year after Friedrich Wilhelm Bessel (1784-1846) published his distance measurements for another star (61 Cygni) in 1838. Consequently Henderson became only the second person to announce actual measurements of distances to a star. He claimed that Alpha Centauri is 3.25 light years away, which has now been superseded by a more accurate value of 4.3 light years. Bessel's slightly more accurate readings showed a distance of 10.4 light years to 61 Cygni. This distance

has been corrected to 11.4 light years using current technology.

Henderson also discovered that Alpha Centauri was in fact a double star, Alpha Centauri A and B. These two stars orbit around each other due to their mutual gravitational pull. This double star was believed to be the star closest to our solar system until 1915 when Robert Innes (1861-1933) discovered a third, smaller red dwarf star within the Alpha Centauri system (see Fig. 5). He called this third, smaller star Proxima Centauri, where “proxima” means “nearest to” in Latin. It has a distance of 4.2 light years from our solar system which indeed makes it the star closest to Earth (other than the Sun, of course).

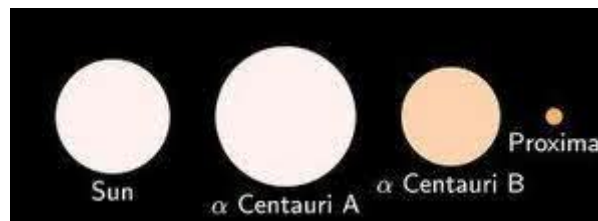


Figure 5: The Alpha Centauri Stars [17].

6 Masses of the Earth, Moon, Sun and Stars

Once the radius r of the Earth is known, its mass M can be calculated using Newton’s Law of Gravitation [18],

$$F_{\text{grav}} = mg = \frac{GMm}{r^2}, \quad (10)$$

where m denotes a test mass, $g = 9.81 \text{ m/s}^2$ the acceleration due to gravity and G Newton’s constant. Once the latter was determined by Cavendish one could determine M by rearranging (10),

$$M = \frac{gr^2}{G} = 5.9742 \times 10^{24} \text{ kg}. \quad (11)$$

The mass of the Moon can be found analogously once the Moon’s surface gravity is known, $g_{\text{moon}} = 1.62 \text{ m/s}^2$ [7]. The latter (a sixth of the Earth value) was measured from spacecrafts and famously experienced by the astronauts from the Apollo space missions. Applying the formula (11) with g replaced by g_{moon} determines the value for the mass of the Moon as $7.3 \times 10^{22} \text{ kg}$.

An alternative method for the determination of masses is based on Kepler’s third law which states that the square of a planetary period T is proportional to the cube of the semi-major axis a of the elliptic orbit. Approximating orbits by circles, a is simply the

orbit radius, i.e. the distance of the planet from the sun. Denoting planetary and solar masses by m_1 and m_2 , respectively, Kepler's 3rd law becomes [19]

$$T^2 = \frac{4\pi^2}{G(m_1 + m_2)} a^3 . \quad (12)$$

Thus, if one of the masses is known together with period T (easily observed) and distance a (see previous sections), the other mass is easily determined. Taking m_1 to be the Earth's mass (11), the (current) mass of the Sun is estimated at 2×10^{30} kg which is roughly 3×10^5 times the mass of the Earth. As stated before, this mass will decrease as fusion reactions convert hydrogen into helium releasing huge amounts of energy in the process [20]. From Einstein's famous relation, $E = mc^2$, we know that this energy loss is equivalent to a loss in mass.

To determine the masses of stars is more difficult. For star *systems* such as Alpha Centauri, however, the methods of this section are applicable. In this way one finds, for instance, that Proxima Centauri has a mass of 1/8 solar masses [21].

7 Discussion and Conclusion

This report has reviewed that the circumference of the Earth has been known accurately (to within a few %) for over two thousand years, an achievement made possible by the geometrical expertise of the ancient Greeks. The mass of the Earth, however, being a non-geometrical quantity, was not found until Newton developed his dynamical Law of Gravitation and Henry Cavendish determined the gravitational constant with his torsion scales.

Although advanced technology was not available in ancient times, the distance of the Moon was determined by Aristarchus, again based on geometrical considerations, with an error of a few percent. The distance to the Sun, on the other hand, was vastly underestimated.

Finding the distance to nearby stars by geometrical means (the parallax) turned out to be the hardest problem. The first measurement of the distance to the closest nearby star (other than the Sun) was made in 1833, published in 1839, and (from hindsight) was off by about 30%. Since then much of the methodology has remained the same but could be significantly improved. This has provided us with more accurate figures about distances of stars and continues to show the efficiency of simple geometry for making astronomical distance measurements (if the stellar objects are not too far away).

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