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The Bending of Beams and the Second Moment of Area

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Abstract
We present an overview of the laws governing the bending of beams and of beam theory. Particular emphasis is put on beam stiffness associated with different cross section shapes using the concept of the second moment of area.

1 Historical Introduction

Beams are an integral part of everyday life, with beam theory involved in the development of many modern structures. Early applications of beam practice to large scale developments include the Eiffel Tower and the Ferris Wheel. Beam theory also was crucial for the Second Industrial Revolution [1].

The theory of beams has a long history. It was believed that Galileo Galilei made the initial attempts at a theory, but then, in 1967, came the discovery of *The Codex Madrid*. This document is written proof that Leonardo da Vinci's published work of 1493 had not only preceded Galileo's by over 100 years, but also represented an improvement. Unlike Galileo, da Vinci correctly identified the stress and strain distribution across a section in bending. Even with this accurate identification, there was still no mention of assessing the strength of a beam once the dimensions are known, and the tensile strength of the material it was made of [2].

This was addressed by Galileo, but due to some incorrect assumptions, Galileo's result was three times larger than the correct value, so it needed considerable improvement. Alternative approaches were published in 1686 and 1713 by Edme Mariotte and Antoine Parent, respectively, but both did not have a serious impact [2].

The major breakthrough came when Daniel Bernoulli and Leonard Euler adapted work previously attempted by Daniel's uncle, Jacob Bernoulli, into what is now known as the *Euler-Bernoulli Beam Theory*. Daniel Bernoulli suggested to Euler that he should apply variational calculus when deriving the differential equations of elastic curves, which he then integrated to obtain the formula for the deflection d of the end of a cantilever,

$$d = \frac{PL^2}{3C} . \tag{1}$$

The constant C is called the *absolute elasticity* and P is the load placed on the cantilever of length L. Bernoulli was also the first to derive the differential equation governing lateral vibrations of prismatic bars, and he made a series of verifying experiments. Euler also considered the buckling of straight bars under axial load, and derived the equation

$$P = \frac{C\pi^2}{4l^2} , \tag{2}$$

where l denotes the position of the load on the beam. He also successfully derived equations for the vibration of beams.

By 1750, Euler had derived many equations for Beam Theory, including those stated above, and his methods could also be extended to the calculation of bending stresses, and hence the bending capacity of beams made of brittle materials [3].

Despite all the work by Euler and Bernoulli, they still did not take into account shear deformation, where the beam changes in shape but not in length, and rotational inertia effects, a property of rotating bodies that defines its resistance to a change in angular velocity about an axis of rotation [4]. This was eventually covered by the *Timoshenko Beam Theory*, developed by Ukrainian-born scientist and engineer Stephen

Timoshenko early in the 20th century [5]. His advanced theory, however, is beyond the scope of this short overview.

2 Laws Governing the Bending of Beams

2.1 Euler-Bernoulli

The Euler-Bernoulli beam theory is the most widely used theory and can be derived from four key principles [6]. These are: kinematics, the constitutive law, resultants and the equilibrium law. By looking at these in detail, we can see exactly how the Euler-Bernoulli beam theory is derived.

2.1.1 Kinematics

This is the method used to measure the displacement of a bending beam from its initial starting place at any given moment in time.

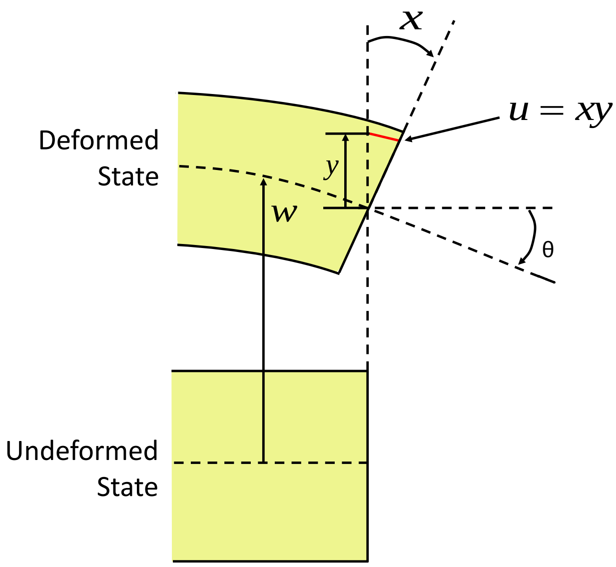

Figure 1: Initial kinematics of a bending beam [7].

In Figure 1 pressure is being applied on the bottom side of the beam. w represents the centre of the beam and θ the angle of rotation of the neutral plane. Similarly χ represents the rotation of the beam's cross section; this is what we want to calculate. u is the displacement of the cross section with respect to the x-axis. Note that we are only considering the beam to bend in the x-axis, and so y is not important for us in this case. A way to model the beam is to consider it being composed of fibres. When the beam bends, the side with the force being exerted on will contract whilst, to compensate, the opposite side of the beam elongates. To model this we consider the strain throughout the entire beam to balance the fact that some parts may contract more than others elongate. Hence we model the strain ϵ as

$$\epsilon = \frac{\mathrm{d}u}{\mathrm{d}x}. \tag{3}$$

Continuing to model only with respect to x, we follow Kirchhoff's assumptions that beam normals remain straight, unstretched and at right angles to the normal plane.

From the third of these assumptions, we see that displacement u and rotation angle χ are related as

$$u(x, y) = \chi(x)y \ , \tag{4}$$

so that the strain (3) becomes

$$\epsilon = \frac{\mathrm{d}\chi}{\mathrm{d}x} y \tag{5}$$

On the other hand, by the definition of the derivative, we have that

$$\chi = -\theta = -\frac{\mathrm{d}w}{\mathrm{d}x} \tag{6}$$

From (5) and (6), we can write ϵ in the form

$$\epsilon = -\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} y \tag{7}$$

2.1.2 Constitutive Law

The constitutive law provides a relation between direct stress and strain (σ and ϵ, respectively) within the beam. "Direct" here means normal to the cross section of the beam. In general, direct stress will be distributed across the beam cross section (chosen to be the yz plane) as shown in Figure 2, so $\sigma = \sigma(x, y)$.

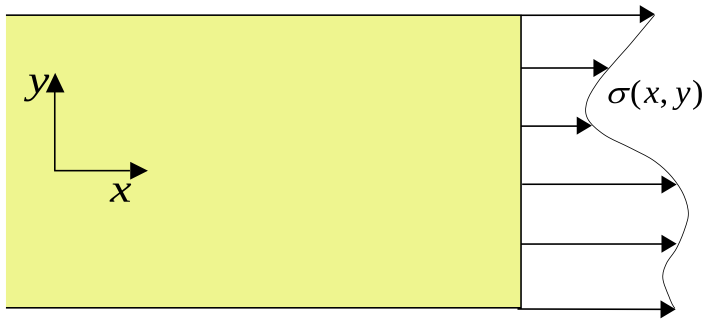

Figure 2: Direct stress distribution across a beam [8].

Young's elastic modulus, E, is a material constant characterising the elasticity of a given material. Mathematically, it is the constant of proportionality relating direct stress and strain in Hooke's law, which in our case reads [8]:

$$\sigma(x, y) = E\epsilon(x, y) . \tag{8}$$

In addition to direct or normal stress σ, there is also shear stress σ_{xy} to be taken into account. The latter acts tangentially to surfaces (see Figure 3).

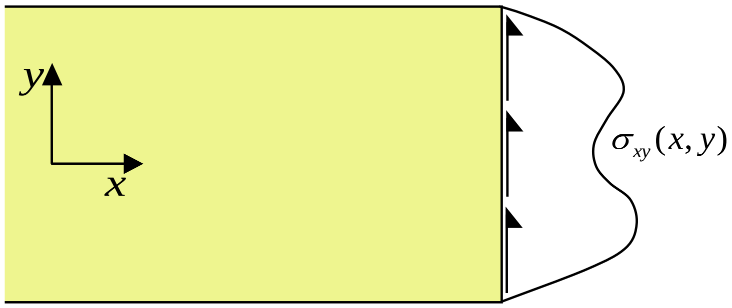

Figure 3: Shear stress distribution across a beam [9].

2.1.3 Resultants

Given the distributions of direct and shear stress we can form moments by integration. These are commonly referred to as resultants. Two important examples are the moment resultant obtained from direct stress,

$$M(x) = \int\int y\sigma(x, y)\mathrm{d}y\mathrm{d}z , \tag{9}$$

and the force resultant, the area integral of shear stress,

$$V(x) = \int\int \sigma_{xy}(x, y)\mathrm{d}y\mathrm{d}z . \tag{10}$$

2.1.4 Equilibrium

Given any section of a beam we choose to work with we can now trade the stresses in both coordinate directions for the resultant forces to eliminate the dependence on the y coordinate.

Taking the derivatives of (9) and (10) with respect to the remaining variable x we find the equilibrium equations

$$\frac{\mathrm{d}V}{\mathrm{d}x} = -p , \tag{11}$$

which introduces the pressure p, and

$$\frac{\mathrm{d}M}{\mathrm{d}x} = -V \,, \tag{12}$$

which relates moment and force resultant, (9) and (10).

The equilibrium conditions (11) and (12) are illustrated in Figure 4.

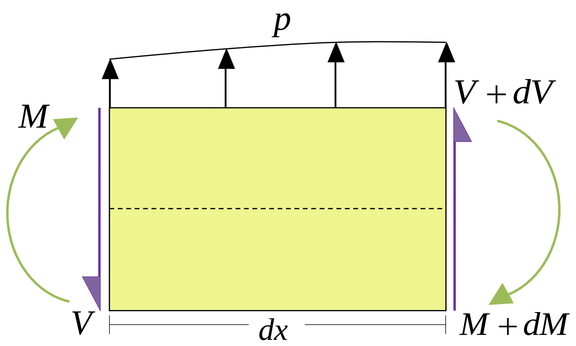

Figure 4: The equilibrium of a section in a beam [10]

2.1.5 Derivation

To obtain the classic Euler-Bernoulli beam equation we first eliminate V from the equilibrium equations (11) and (12), which gives us

$$\frac{\mathrm{d}^2M}{\mathrm{d}x^2} = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\int\int y\sigma(x,y)\mathrm{d}y\mathrm{d}z = p \,, \tag{13}$$

with the double integral extending over the beam cross section (which may depend on position x). Making use of the constitutive law (8) this becomes

$$E\frac{\mathrm{d}^2}{\mathrm{d}x^2}\int\int y\epsilon(x,y)\mathrm{d}y\mathrm{d}z = p \,. \tag{14}$$

The y dependence of ϵ is given by the kinematics (5) so that (14) turns into

$$E\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left\{\frac{\mathrm{d}\chi}{\mathrm{d}x}\int\int y^2\,\mathrm{d}y\mathrm{d}z\right\} = p \,. \tag{15}$$

Eliminating the angle χ via (6) and denoting the double integral (the area moment of inertia, see next section) by I_x, we finally obtain the Euler-Bernoulli static beam equation [6, 11],

$$E\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left\{I_x\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\right\} = -p \,. \tag{16}$$

This is a fourth order differential equation for the locus $w(x)$ of the beam centre as a function of position x along the beam.

3 Beam Stiffness

In this section we compare the stiffness of beams with different cross sections. Clearly, the shape of a beam's cross section is fundamental in determining how much the beam will deflect under load. The mathematics relating to this deflection is presented through the concept of the 'second moment of area'. This can be calculated for different beam cross sections to obtain an idea of the rigidity. We have chosen two common beam cross section shapes, and we discuss the stiffness of each one. Finally, we will be able to make a decision as to which shape would be more resistant to bending at a certain cross sectional area.

3.1 Second Moment of Area

The second moment of area is known by several different names, including the area moment of inertia, the moment of inertia of plane area and the second moment of inertia. It is a property of a cross section that can be used to predict the resistance of beams to bending and deflection. The deflection of a beam under load depends not only on the load, but also on the geometry of the beam's cross section. The second moment of area has units of *length* to the fourth power and, therefore, has SI units of m^4. Beams with a large second moment of area are more resistant to bending, so are stiffer than those with a small second moment of area. This is why beams with a higher second moment of area, such as I-beams, are often seen in the construction of buildings. The second moment of area is calculated using the following equations [12],

$$I_x = \int y^2 \mathrm{d}A \,, \tag{17}$$

$$I_y = \int x^2 \mathrm{d}A \,, \tag{18}$$

where x and y are the coordinates of an infinitesimal area element $\mathrm{d}A$, as seen in Figure 5. When using these equations it is important to note that the cross sectional area is parallel to the xy plane while the z-axis runs along the length of the beam being bent. Typically the beam is loaded along the z-axis with the forces acting in the vertical direction, along the y-axis. With this type of loading, the second moment of area about the x-axis is required; therefore the formula for I_x should be used. Sometimes a beam may be loaded laterally with loads parallel to the x-axis. In this case, the formula for I_y should be used to find the second moment of area about the y-axis.

3.2 Beam Cross Sections

Having defined the crucial factors associated with beam stiffness and the mathematical method used to rate a beam's deflection under load, we can look into some

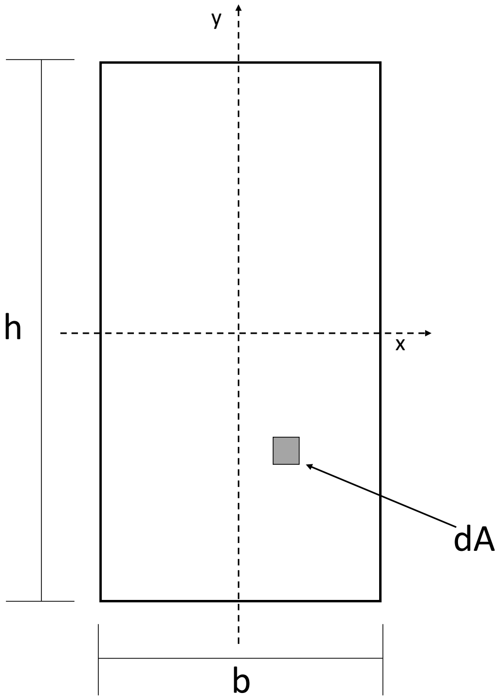

Figure 5: Rectangular beam cross section centred at the origin with height, *h* and width, *b*.

common beam cross sectional shapes.

3.2.1 Solid Square Cross Section

An extremely common cross sectional shape for a beam is a solid square or, the geometrically similar, rectangle. This is possibly the simplest beam to manufacture and provides many uses at a relatively low cost. We want to address the stiffness associated with this beam shape. As explained before, we can do this by looking at the second moment of area. Figure 5, shows the position of a rectangle, centred at the origin, with the horizontal x-axis and vertical y-axis running through the middle of the shape [13]. Clearly, for a square cross section the height, h, would be equal to the width, b.

We want to find out how the cross sectional area affects the deflection in the y-direction as a load is applied along the z-axis with the force in the vertical direction. We therefore need formula (17) for I_x. Then, choosing the appropriate integration limits and identifying $\mathrm{d}A = \mathrm{d}x\mathrm{d}y$, we have

$$I_x = \int y^2 \mathrm{d}A = \int_{-h/2}^{h/2} y^2 \mathrm{d}y \int_{-b/2}^{b/2} \mathrm{d}x = \frac{bh^3}{12} \,. \tag{19}$$

This is the equation for the second moment of area for a beam with a solid rectangular cross section. As expected, it is dependent on the height, h and the width, b. This also leads to the formula for the second moment of area of a beam with a square cross section by setting the dimensions h and b equal to each other. Straight away, we can draw an obvious conclusion: a beam with a large rectangular cross sectional area (i.e. large h and b) will have a higher second moment of area and will therefore be stiffer than a beam with a small rectangular cross sectional area. As the height is the dominant term in the formula, we can see that varying this will have a greater impact on the value of the second moment of area.

3.2.2 Solid Circular Cross Section

We now move on to look at a beam with a different cross sectional shape, a circle. Again, we can use the concept of the second moment of area to gain an idea of the stiffness achieved by using a circular beam. Using this, we want to compare the stiffness of beams with the same cross sectional areas but one of a square cross section, the other with a circular. The circle is centred about the origin with radius, a, as shown in Figure 6.

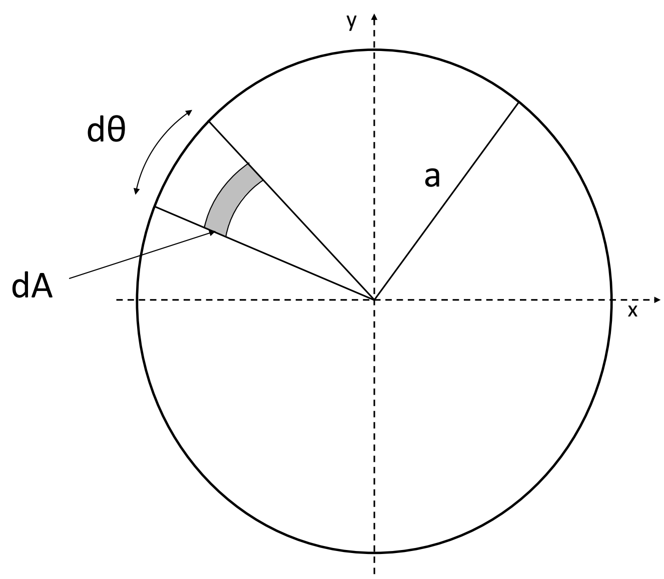

Figure 6: Circular beam cross section with radius a. Also shows the element $\mathrm{d}A$.

Again, we are looking at the deflection in the vertical direction so need the I_x from (17). As we are dealing with a circle it is easier to calculate the integral using polar coordinates where the surface element is $\mathrm{d}A = r\mathrm{d}r\mathrm{d}\theta$ (cf. Figure 6), and $y = r\sin\theta$.

Armed with this information we are able to tackle the integral,

$$I_x = \int y^2 \mathrm{d}A = \int_0^{2\pi} \sin^2\theta \mathrm{d}\theta \int_0^a r^3 \mathrm{d}r = \frac{r^4\pi}{4} . \tag{20}$$

The second moment of area, on its own, does not provide too much information, only that a thicker beam will result in a higher second moment of area and therefore is more resistant to bending. This is obvious in day to day life. The real power of area moments shows up in the comparison of different cross sectional shapes.

3.3 Square and Circular Beam Cross Sections: Comparison

As stated, to determine which cross sectional shape beam provides more resistance to bending, the values of the second moment of area need to be compared. So far in this section we have derived the formulae for calculating the second moment of area for a beam with a solid square or rectangular cross section and a beam with a solid circular cross section. We now want to determine which of these shapes gives a stiffer beam at a given cross sectional area, $A = b^2$. The respective second moments of area are

$$I_\square = \frac{b^2}{12} = \frac{A^2}{12} , \tag{21}$$

$$I_\circ = \frac{r^4\pi}{12} = \frac{A^2}{4\pi} < I_\square , \tag{22}$$

the latter inequality following from $4\pi > 12$. Thus, we can clearly see that the second moment of area for a beam with a circular cross section is slightly smaller than that of a beam with a square cross section of the same area, the relative difference being about 5 %. Therefore, the square beam provides slightly more resistance to bending than the circular one. This shows how engineers can straightforwardly compare the rigidity of beams with various cross sections.

4 Discussion and Conclusion

As we have seen, the derivation of the Euler-Bernoulli beam theory from initial principles provides an accurate representation of the pressure on a static beam given its elasticity, length and position. Timoshenko then improved on this, with the consideration of shear deformation and rotational inertia effects. For the Euler-Bernoulli beam theory, inertia (the second moment of area) plays a key role. Its main ingredient is the cross sectional shape of the beam. We have shown mathematically that a square cross section provides more resistance to bending than a circular one. This is unsurprising as rectangular shapes are the more popular choice in construction if a lateral load is to be held (Fig. 7). The general rule for increasing rigidity is to make the area moments large. This can be done most elegantly (keeping beam weight within reasonable limits) by having cross sectional area concentrated away from the beam centre. The next logical step would thus be to look at more complicated cross sections used in construction, such as hollow beams, I-beams or T-sections.

Figure 7: A typical use of square beams [14].

References

[1] http://en.wikipedia.org/wiki/Euler-Bernoulli_beam_theory, accessed 20-01-2013.

[2] http://newtonexcelbach.wordpress.com/2008/02/27/the-history-of-the-theory-of-beam-bending-part-1/, accessed 20-01-2013.

[3] http://newtonexcelbach.wordpress.com/2008/07/12/the-history-of-the-theory-of-beam-bending-part-3/, accessed 20-01-2013.

[4] http://en.wikipedia.org/wiki/Moment_of_inertia, accessed 20-01-2013.

[5] http://en.wikipedia.org/wiki/Timoshenko_beam_theory, accessed 20-01-2013.

[6] http://www.efunda.com/formulae/solid_mechanics/beams/theory.cfm, accessed 20-01-2013.

[7] http://www.efunda.com/formulae/solid_mechanics/beams/theory_kinematics.cfm, accessed 20-01-2013.

[8] http://www.efunda.com/formulae/solid_mechanics/beams/theory_constitutive.cfm, accessed 20-01-2013.

[9] http://www.efunda.com/formulae/solid_mechanics/beams/theory_resultants.cfm, accessed 20-01-2013.

[10] http://www.efunda.com/formulae/solid_mechanics/beams/theory_equilibrium.cfm, accessed 20-01-2013.

[11] P. K. Kythe and D. Wei, *An Introduction to Linear and Nonlinear Finite Element Analysis: A Computational Approach*, Springer, 2004.

[12] http://efunda.com/math/areas/IndexArea.cfm, accessed 20-01-2013.

[13] http://youtube.com/watch?v=QR5mivx0xc0, accessed 20-01-2013.

[14] http://images01.olx.com/ui/10/41/83/1292201702_146163883_8-Reclaimed-Barn-Beams-by-JLT-Millworks-Lumber-Jane-Janice-Tupper–1292201702.jpg, accessed 20-01-2013.